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Second-harmonic wave diffraction at large depths

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An approximation of the second-order diffraction potential is derived, for water waves of small amplitude incident upon a fixed body in a fluid of large depth. Attention is focused on the second-harmonic component of this potential, in terms of the fundamental incident-wave frequency, and on the particular solution of the inhomogeneous free-surface boundary condition with quadratic forcing by the first-order solution. By considering only the far-field approximation of the forcing function, a simple solution is derived in the near field of the body which is dominant when the submergence of the field point is large. The validity of this approach is confirmed by comparisons with two-dimensional experiments and three-dimensional computations.

1. Introduction

If water waves of small amplitude are incident upon a diffracting body, the resulting hydrodynamic pressure force exerted on the body includes a first-order component linear in the amplitude of the incident waves, and higher-order nonlinear components which may be derived formally by a perturbation expansion. Notwithstanding the assumptions of this classical approach, applications exist where the significance of the second-order force is comparable with or even greater than the linear component.

For excitation at a single frequency ω the second-order pressure and the resulting force acting on the body consist of mean components which are independent of time, and second-harmonic components which are oscillatory at the frequency 2ω. In a wave spectrum these two forces are generalized, and known respectively as the 'slowly varying' or 'difference-frequency' drift forces, and the 'sum-frequency' forces. These have obvious relevance in applications where the body is resonant at long or short periods, respectively, relative to the fundamental period of the incident waves.

Recent computations for axisymmetric bodies by Eatock Taylor & Hung (1987), Shimada (1987), and Kim & Yue (1988, 1989) reveal that the second-harmonic component of the diffraction pressure field also is particularly significant at large depths. For large offshore structures such as tension-leg platforms, which are restrained by taut moorings, the resulting contribution to the vertical pressure force and mooring loads may dominate the first-order linear component. Experimental evidence of the same phenomenon is presented by Johansson (1989), who studies two-dimensional floating breakwaters and notes that 'the vertical force is significantly contaminated by a double harmonic'.

Unlike the second-order mean drift force, which can be evaluated consistently from the first-order velocity potential, the second-harmonic force depends in part on the solution for the second-order potential. The principal complication of the corresponding boundary-value problem, relative to its first-order counterpart, is the inhomogeneous free-surface boundary condition. This results from quadratic forcing by the first-order solution, and can be interpreted physically as an imposed pressure on the free surface. In the diffraction problem, where first-order plane progressive waves are incident from infinity, the quadratic forcing function persists in the far field, complicating the formulation and solution of the second-order potential.

The complications associated with the second-order inhomogeneous free-surface condition have motivated some incomplete solutions of the second-order diffraction problem, in which the potential satisfies a homogeneous linear free-surface condition. The practical defects of these results are emphasized by Kim & Yue (1988), who show from an elaborate numerical solution, restricted to axisymmetric structures, that the dominant component of the second-harmonic vertical force is associated directly with the particular solution of the inhomogeneous free-surface condition.

A useful analogy can be inferred from the simpler analysis of two-dimensional deep-water standing waves, where the second-order solution of the free-surface condition includes a second-harmonic pressure component independent of depth (cf. Wehausen & Laitone 1960, pp. 665–666). The same phenomenon occurs for a partial standing wave, resulting from wave reflection by a two-dimensional body (Ogilvie 1983). Since the transmitted wave lacks such a component, a transition must occur in the horizontal direction, at large depths, between the $O(1)$ pressure far from the body in the 'upstream' reflected-wave direction, and the exponentially small pressure in the 'downstream' transmitted-wave direction. This suggests that the second-harmonic pressure in the vicinity of the diffracting body may be more important at large depths than the first-order component.

A similar result can be expected in three dimensions, where the inhomogeneous free-surface boundary condition forces a non-radiating second-order solution along the ray directly opposite to the propagation angle of the incident wave. This singular feature of the far-field asymptotic solution was noted by Molin (1979), and has been discussed extensively in connection with the appropriate statement of the second-order radiation condition (Mei 1983; Molin 1986).

It is reasonable to assume that a connection exists between the far-field behaviour of the quadratic forcing function in the second-order free-surface condition, the slow rate of attenuation with depth of the resulting near-field solution, and the large second-harmonic forces observed in experiments and computations. Moreover, if this assumption is correct, it suggests that an analysis which considers only the far-field forcing function may lead to a relatively simple second-order solution which approximates the dominant second-order pressure field acting on the body.

In the present work an asymptotic solution is developed to establish this connection. Assuming that the submergence of the field point beneath the free surface is sufficiently large to neglect the components of the fluid pressure which depend exponentially on the vertical coordinate, an asymptotic expansion is derived for the second-order potential in inverse powers of the vertical coordinate z. The leading term in this expansion is independent of z in two dimensions, and of order z^{-1} in three dimensions.

The basic analysis is first developed in §2 for two-dimensional diffraction of an incident plane progressive wave system by a fixed body, in a fluid of infinite depth. The corresponding three-dimensional analysis is presented in §3. Finite-depth effects are estimated in §4, and in §5 the analysis is extended to include the sum-frequency force components in a pair of incident waves with different frequencies. To demonstrate the practical validity of this approach comparisons are made in §6 with the two-dimensional experiments of Johansson (1989), and with the three-dimensional axisymmetric numerical results of Kim & Yue (1988).

2. Two-dimensional analysis

If plane progressive waves of small amplitude A are diffracted by a fixed body, the time-harmonic components of the velocity potential can be expanded in the form

$$\Phi(\boldsymbol{x},t) = \mathrm{Re}\,(\phi_1(\boldsymbol{x})\,\mathrm{e}^{\mathrm{i}\omega t}+\phi_2(\boldsymbol{x})\,\mathrm{e}^{2\mathrm{i}\omega t}+\dots), \tag{1}$$

where $\phi_n = O(A^n)$. Each term in this expansion is governed by Laplace's equation in the fluid domain, and satisfies a homogeneous Neumann boundary condition on the body surface.

The first-order diffraction potential ϕ_1 can be expressed in the form $\phi_1 = \phi_1^{\mathrm{I}}+\phi_1^{\mathrm{S}}$, where

$$\phi_1^{\mathrm{I}} = \frac{gA}{\omega}\mathrm{e}^{Kz-\mathrm{i}Kx} \tag{2}$$

is the incident-wave potential and ϕ_1^{S} denotes the scattered potential. Here $K = \omega^2/g$ is the wavenumber, $z = 0$ is the plane of the free surface with the $+z$-axis directed upwards, and the $+x$-axis is in the direction of incident-wave propagation. Both components of ϕ_1 satisfy the linear free-surface boundary condition $\phi_{1z}-K\phi_1 = 0$ on $z = 0$, and vanish for $z\rightarrow-\infty$.

Including terms of second order, the free-surface boundary condition can be expressed in the form

$$\Phi_{tt}+g\Phi_z = -2\boldsymbol{\nabla}\Phi\cdot\boldsymbol{\nabla}\Phi_t+\frac{1}{g}\Phi_t(\Phi_{ttz}+g\Phi_{zz}) \quad (z=0) \tag{3}$$

(cf. Newman 1977, equation 6.33). Substituting (1) and retaining only the second-harmonic terms gives

$$\phi_{2z}-4K\phi_2 = q(x), \tag{4}$$

where the 'quadratic forcing function' q is defined in terms of the first-order solution by

$$q = \left[\frac{-\mathrm{i}\omega}{g}\boldsymbol{\nabla}\phi_1\cdot\boldsymbol{\nabla}\phi_1+\frac{\mathrm{i}\omega}{2g}\phi_1(\phi_{1zz}-K\phi_{1z})\right]_{z=0}. \tag{5}$$

In the remainder of this Section we assume two-dimensional motions which are independent of the coordinate y, as would be the case if the diffracting body is cylindrical with its axis perpendicular to the plane x–z.

A particular solution $\phi_2(x,z)$ can be derived formally by interpreting $q(x)$ as an oscillatory pressure imposed on the free surface, with the result

$$\phi_2 = \frac{1}{\pi}\int_{-\infty}^{\infty} q(\xi)\,\mathrm{d}\xi\int_0^{\infty}\frac{\cos k(x-\xi)\,\mathrm{e}^{kz}}{k-4K}\mathrm{d}k \tag{6}$$

(cf. Wehausen & Laitone 1960, equation 21.21). Here the first integral is over the domain of the free surface, excluding the portion occupied by the body. In the second integral the contour of integration is deformed above the pole in the complex k-plane, to conform with the radiation condition.

The complete second-order potential includes, in addition to (6), a solution of the homogeneous free-surface boundary condition which cancels the normal velocity on the body induced by (6). This 'homogeneous' component of the second-order solution is similar in form to the first-order scattered potential, except that the wavenumber K is replaced in the free-surface condition by $4K$. It follows that the effects of the homogeneous component are confined to an exponentially small layer near the free surface, and for this reason it is not significant in the analysis to follow.

An asymptotic expansion of (6) can be derived for large (negative) values of the vertical coordinate z, by expanding the factor $1/(k-4K)$ in powers of k and integrating term-by-term. Neglecting contributions which are exponentially small, it follows that

$$\phi_2 \sim \frac{1}{4\pi K}\sum_{m=0}(4K)^{-m}\frac{\partial^m}{\partial z^m}\int_{-\infty}^{\infty}\frac{z}{z^2+(x-\xi)^2}q(\xi)\,\mathrm{d}\xi, \tag{7}$$

where the transform

$$\int_0^{\infty}\cos(ku)\,\mathrm{e}^{-kv}\,\mathrm{d}k = v/(u^2+v^2)^{\frac{1}{2}}$$

has been used.

Equation (7) provides a representation of the particular solution (6) in terms of vertical dipoles and higher-order multipoles on the free surface. For large values of $|z|$ the terms in the integrand are $O(z^{-(m+1)})$; if the integral of $q(x)$ over the complete free surface is finite, the leading term in (7) is $O(z^{-1})$. In this context the far-field behaviour of the quadratic forcing function is critical.

Far downstream, as $x\rightarrow+\infty$, the first-order solution is a monochromatic transmitted wave with no second-order component. It is easy to confirm that (5) vanishes in this limit. On the other hand as $x\rightarrow-\infty$ the first-order solution is a partial standing wave of the form

$$\phi_1 \sim \frac{gA}{\omega}(\mathrm{e}^{Kz-\mathrm{i}Kx}+\tilde{R}\,\mathrm{e}^{Kz+\mathrm{i}Kx}), \tag{8}$$

where $\tilde{R}$ is the reflection coefficient. Substituting this result into (5) it follows that

$$q \sim -4\mathrm{i}\omega KA^2\tilde{R} \equiv -4KC \quad (x\rightarrow-\infty) \tag{9}$$

where C is a constant. From the free-surface condition (4), C is equivalent to the constant part of the second-order potential far upstream, i.e. the second-order partial standing wave.

Neglecting a local component which vanishes at infinity in both directions, the quadratic forcing function can be represented by

$$q(x) = -4KCH(-x), \tag{10}$$

where $H(x)$ denotes the Heaviside unit step function. Substituting this result in (7) and integrating over the extended domain of the free surface, including any portion interior to the body, gives the leading contribution from the dipole term ($m=0$) in the form

$$\phi_2 \sim -\frac{C}{\pi}\int_{-\infty}^{0}\frac{z}{z^2+(x-\xi)^2}\mathrm{d}\xi = \frac{C}{2}\left[1-\frac{2}{\pi}\tan^{-1}\left(\frac{x}{|z|}\right)\right]. \tag{11}$$

The error due to neglecting the local component of q can be estimated in the following manner. The first-order potential can be approximated for large values of $|x|$ in terms of elementary plane waves plus a local (evanascent) component. The latter is asymptotic to a vertical Rankine dipole, of order (x^{-2}) in the far field. Thus the integral of the local component of q over the entire free surface is finite, and the error from neglecting this contribution in (7) is $O(z^{-1})$. In the absence of a more complete analysis which evaluates the local component, there is no point in including the higher-order terms ($m \geqslant 1$) in (7).

Equation (11) can be interpreted as the potential due to a vortex at the origin. The constant is such that this potential vanishes far downstream, where there is a progressive transmitted wave. Far upstream (11) is asymptotic to the constant C associated with the partial standing wave. For field points near the diffracting body, such that $|z| \gg |x|$, the potential ϕ_2 is precisely the average of the two far-field limits. For sufficiently large depths this second-order potential will dominate the first-order component.

3. Three-dimensional analysis

For a three-dimensional body of arbitrary form, the first-order scattered potential satisfies the far-field radiation condition

$$\phi_1^S \sim Af(\theta)\,(Kr)^{-\frac{1}{2}}e^{Kz-iKr} \quad (Kr \gg 1). \tag{12}$$

Here r, θ are polar coordinates about the vertical axis, and $f(\theta)$ describes the angular dependence of this potential in the far field.

The quadratic forcing function q includes contributions from cross-products of the incident and scattered potentials, and products of the scattered potential with itself. (For the case of infinite depth there is no contribution from the incident wave alone.) In the far field, where (12) is valid, the cross-product component is $O((Kr)^{-\frac{1}{2}})$, whereas products of the scattered potential with itself are $O((Kr)^{-1})$. (In fact, the latter component in (5) vanishes to this order, and the leading-order contribution is $O((Kr)^{-2})$.) Thus the dominant forcing function in the far field is associated with cross-terms between the incident and scattered potentials. Only the first term on the right-hand side of (5) contributes when (2) and (12) are substituted, and the far-field approximation of the quadratic forcing function is defined in polar coordinates as

$$q(r,\theta) \sim -2iK^2A^2f(\theta)\,(1-\cos\theta)\,(Kr)^{-\frac{1}{2}}e^{-iKx-iKr} = F(\theta)\,(Kr)^{-\frac{1}{2}}e^{-iKr(1+\cos\theta)}. \tag{13}$$

Here $F(\theta) = -2iK^2A^2f(\theta)\,(1-\cos\theta)$, and $\theta = 0$ is the direction of incident-wave propagation. Note that $F(0) = 0$, in accordance with the fact that the scattered wave is purely progressive along the ray $\theta = 0$, whereas the forcing function is non-zero and non-oscillatory along the ray $\theta = \pi$. The error in (13) is a factor $1+O((Kr)^{-1})$.

A particular solution analogous to (2) can be derived using Fourier–Bessel transforms, in the form

$$\phi_2 = \frac{1}{2\pi}\iint q\,dS \int_0^\infty \frac{k}{k-4K}e^{kz}J_0[k(r^2+\rho^2-2r\rho\cos(\theta-\alpha))^{\frac{1}{2}}]\,dk \tag{14}$$

(cf. Wehausen & Laitone 1960, equation 21.4, with a correction in sign). Here J_0 is the Bessel function of the first kind, and the contour in the single integral is deformed to pass above the pole. The surface integral is over the free surface, i.e. the portion of the plane $z = 0$ exterior to the body.

Proceeding as in §2, an asymptotic expansion of (14) can be derived by expanding the factor $k/(k-4K)$ in powers of k and integrating term by term. Neglecting contributions which are exponentially small,

$$\phi_2 \sim -\frac{1}{2\pi}\sum_{m=1}(4K)^{-m}\frac{\partial^m}{\partial z^m}\iint [R^2+\rho^2-2r\rho\cos(\theta-\alpha)]^{-\frac{1}{2}}q\,dS, \tag{15}$$

where $R = (r^2+z^2)^{\frac{1}{2}}$ denotes the spherical radius and the transform

$$\int_0^\infty J_0(ku)\,\mathrm{e}^{-kv}\,\mathrm{d}k = (u^2+v^2)^{-\frac{1}{2}}$$

has been used.

Equation (15) provides a representation of the particular solution (14) in terms of vertical dipoles and higher-order multipoles on the free surface. For large values of $|z|$ the terms in this expansion are $O(z^{-m})$. The leading-order contribution, of order z^{-1}, is

$$\phi_2 \sim \frac{z}{8\pi K}\iint [R^2+\rho^2-2r\rho\cos(\theta-\alpha)]^{-\frac{3}{2}} q\,\mathrm{d}S. \tag{16}$$

If the far-field approximation (13) is substituted in (16) and integrated over the entire plane $z = 0$,

$$\phi_2 \sim \frac{z}{8\pi K^{\frac{3}{2}}}\int_0^\infty \rho^{\frac{1}{2}}\mathrm{d}\rho \int_0^{2\pi} F(\alpha)\,\mathrm{e}^{-\mathrm{i}K\rho(1+\cos\alpha)}\,[R^2+\rho^2-2r\rho\cos(\theta-\alpha)]^{-\frac{3}{2}}\,\mathrm{d}\alpha. \tag{17}$$

The error in this approximation is estimated below, following (20).

If the variable ρ in (17) is rescaled in terms of R, the argument of the exponential function is proportional to the large parameter KR. Thus the second integral in (17) can be evaluated by the method of stationary phase, with the result

$$\phi_2 \sim \frac{F(\pi)\,\mathrm{e}^{\mathrm{i}\pi/4}z}{4(2\pi)^{\frac{1}{2}}K^2}\int_0^\infty \mathrm{d}\rho[R^2+\rho^2+2r\rho\cos\theta]^{-\frac{3}{2}}. \tag{18}$$

There is no contribution from the stationary point at $\alpha = 0$, since $F(0) = 0$. After evaluating the last integral it follows that

$$\phi_2 \sim \frac{F(\pi)\,\mathrm{e}^{\mathrm{i}\pi/4}}{4(2\pi)^{\frac{1}{2}}K^2}\frac{z}{R(R+x)}. \tag{19}$$

For field points close to the vertical axis,

$$\phi_2 \sim \frac{F(\pi)\,\mathrm{e}^{\mathrm{i}\pi/4}}{4(2\pi)^{\frac{1}{2}}K^2}\frac{1}{z}. \tag{20}$$

The error in (17)–(20) may be estimated by considering the error $\tilde{q}$ in (13), and the corresponding contribution $\tilde{\phi}_2$ in (16). The free-surface integral may be decomposed into two parts, separated by a circular partition at a fixed radius $\rho = M$ which is sufficiently large to permit the use of the far-field approximations (12) and (13) when $\rho > M$. In the inner integral $\rho^{\frac{1}{2}}\tilde{q}$ is bounded, and an application of the mean-value theorem indicates that this integral gives a contribution to $\tilde{\phi}_2$ of order $M^{\frac{1}{2}}z^{-2}$. In the outer integral $\tilde{q} = O((Kr)^{-\frac{3}{2}})$, and an analysis similar to (17) and (18) may be performed with the resulting contribution to $\tilde{\phi}_2$ of order $z^{-2}\log(M/z)$. Thus (19) and (20) are consistent asymptotic approximations of the second-order potential.

An alternative to the above analysis follows from (14) by using the addition theorem for the Bessel function (cf. Wehausen & Laitone 1960, equation 21.5), integrating first in the radial direction between $\rho = 0$ and ∞, then using the stationary-phase approximation, and finally approximating the remaining integral in k by Watson's Lemma. This approach leads to the Fourier series

$$\phi_2 \sim -\frac{F(\pi)\,\mathrm{e}^{\mathrm{i}\pi/4}}{4(2\pi)^{\frac{1}{2}}K^2R}\sum_{n=0}^{\infty}\epsilon_n(-)^n\cos n\theta\left(\frac{r}{R-z}\right)^n, \tag{21}$$

where $\epsilon_0 = 1$, and $\epsilon_n = 2$ for $n \geqslant 1$. As $r \to 0$ with $z < 0$, the factor $r/(R-z) \to r/2|z|$. The equivalence of (19) and (21) can be verified directly (cf. Gradshteyn & Ryzhik 1965, equation 1.447(3)). This Fourier series is useful for evaluating the pressure force on an axisymmetric body.

4. Finite depth

For a fluid of constant finite depth h the wavenumber K is modified, the incident-wave potential contains a second-order component, and the second term on the right-hand side of (5) gives a non-vanishing contribution in the far field. However, these modifications are not significant if the depth exceeds about one wavelength. The modification of the last integral in (6) or (14) is more important, with the exponential dependence on the vertical coordinate replaced by hyperbolic functions (cf. Wehausen & Laitone 1960, equation 21.10). The three-dimensional analysis is facilitated by using the alternative approach which leads to (21), and after performing the integration over the free surface it follows that

$$\phi_2 \sim \frac{F(\pi)\,\mathrm{e}^{\mathrm{i}\pi/4}}{(2\pi)^{\frac{1}{2}} K} \sum_{n=0}^{\infty} \epsilon_n (-)^n \cos n\theta \int_0^\infty \frac{\cosh k(z+h)}{k \sinh kh - 4K \cosh kh} J_n(kr)\, \mathrm{d}k. \tag{22}$$

Assuming both the depth h and submergence $|z|$ to be large, the leading-order contribution is from the vicinity of $k = 0$ and the first term in the denominator of the integrand can be neglected. The resulting integral transform is complicated, however, and our attention is restricted to the special case $r = 0$, where

$$\phi_2 \sim \frac{F(\pi)\,\mathrm{e}^{\mathrm{i}\pi/4}}{(2\pi)^{\frac{1}{2}} 4K^2} \int_0^\infty \frac{\cosh k(z+h)}{\cosh kh} \mathrm{d}k = \frac{F(\pi)\,\mathrm{e}^{\mathrm{i}\pi/4}}{4(2\pi)^{\frac{1}{2}} K^2 z} \left[\frac{\pi z}{2h} \operatorname{cosec} \frac{\pi z}{2h}\right] \tag{23}$$

(cf. Gradshteyn & Ryzhik 1965, equation 3.511(4)). The factor in square brackets, which represents the correction to (20) for finite depth, increases from one when $|z| \ll h$ to $\frac{1}{2}\pi$ when $z = -h$.

The effect of finite depth is less important in two dimensions. Following a similar extension with (9) substituted in (6) gives a delta-function $\delta(k)$ for the first integral; since the hyperbolic functions in (22) are irrelevant when $k = 0$, the leading-order solution (11) is not affected by the finite depth.

5. The sum-frequency solution

In second-order spectral analysis it is necessary to consider the interaction between two wave components with frequencies ω_j, ω_l and corresponding wavenumbers K_j, K_l. In addition to second-harmonic terms oscillatory with frequencies $2\omega_j, 2\omega_l$, the second-order potential, pressure, and force include a 'sum-frequency' component with frequency $\omega_j + \omega_l$, as emphasized by Kim & Yue (1988). Assuming $\omega_j \neq \omega_l$, the interaction between the incident and scattered waves is relatively weak in the far field, but as the difference $\omega_j - \omega_l \to 0$ a stronger interaction occurs analogous in the limit to the pure second harmonic.

To analyse the sum-frequency component in three dimensions, assume that the first-order incident-wave potential (2) corresponds to (ω_j, K_j) and the scattered potential (12) to (ω_l, K_l). The far-field free-surface boundary condition (4) is then modified as follows:

$$\phi_{2z} - 4K_{jl}\phi_2 \sim -\mathrm{i}K_j K_l(1+\omega_l/\omega_j)\,A_j A_l\, f_l(\theta)\,(1-\cos\theta)\,(K_l r)^{-\frac{1}{2}}\,\mathrm{e}^{-\mathrm{i}K_j x - \mathrm{i}K_l r}$$
$$= F_{jl}(\theta)\,(K_l r)^{-\frac{1}{2}}\,\mathrm{e}^{-\mathrm{i}K_j r\cos\theta - \mathrm{i}K_l r} \quad (z=0), \quad (24)$$

where $K_{jl} = (\omega_j+\omega_l)^2/4g$, and $F_{jl}(\theta) = -\mathrm{i}K_j K_l(1+\omega_l/\omega_j)A_j f_l(\theta)\,(1-\cos\theta)$.

A solution analogous to (17) can be derived, with the principal modification that the exponential function is replaced by $\exp(-\mathrm{i}\rho(K_j\cos\alpha + K_l))$. The points of stationary phase are unchanged, with the result

$$\phi_2 \sim \frac{F_{jl}(\pi)\,\mathrm{e}^{\mathrm{i}\pi/4} z}{4(2\pi K_j, K_l)^{\frac{1}{2}} K_{jl}} \int_0^\infty \mathrm{d}\rho\, \mathrm{e}^{\mathrm{i}\rho\delta}[R^2+\rho^2+2r\rho\cos\theta]^{-\frac{3}{2}}, \quad (25)$$

where $\delta = (K_j - K_l)$.

In the special case $r=0$ the integral in (25) can be evaluated to give

$$\phi_2 \sim -\frac{F_{jl}(\pi)\,\mathrm{e}^{\mathrm{i}\pi/4}}{(2\pi K_j K_l)^{\frac{1}{2}} 4K_{jl}}\left[|\delta| K_1 + \mathrm{i}\frac{\pi}{2}\delta(L_1 - I_1) + \mathrm{i}\delta\right], \quad (26)$$

where the modified Bessel functions I_1, K_1 and the modified Struve function L_1 are all of argument $|z\delta|$. Note that as $\delta \to 0$, with z fixed, (26) tends to the limiting value (20) for a single wave component, with ϕ_2 of order z^{-1}. On the other hand, if $|z\delta| \gg 1$, the factor in square brackets is of order $z^{-2}\delta^{-1}$. Thus the sum-frequency pressure attenuates more rapidly with depth than the monochromatic second harmonic. This is to be expected, since the dominant second-harmonic component is associated with the coincidence in (13) between the incident and scattered waves, such that the sum of their vector wavenumbers is zero, and this condition cannot be satisfied if the respective frequencies differ.

6. Applications

Comparisons will be made with the two-dimensional experiments of Johansson (1989), and with the axisymmetric three-dimensional computations by Kim & Yue (1988). In both cases we consider only the contributions to the second-order pressure and force associated with the particular solution of the inhomogeneous free-surface condition.

Johansson (1989) reports measurements on a fixed rectangular cylinder. Dimensional data are given for a section of width 1.99 m, draught 0.154 m and beam 0.615 m, in a fluid of depth 0.769 m. Only one data point is given for the second-harmonic vertical force, in waves of period 0.87 s and amplitude $A = 0.024$ m. At this condition the measured force is between 36 and 37 N, or 18–18.5 N/m width. (The indicated range of uncertainty is associated with reading the value from Johansson's figure 5.17, and does not account for estimates of the experimental errors.) From the theory in §2 the same force is calculated by integrating over the cylinder bottom the second-order pressure $p_2 = -2\mathrm{i}\omega\rho\phi_2 \sim \rho\omega^2 A^2 R$. (Since the arctangent in (11) is asymmetric with respect to an origin at the midpoint of the cylinder it does not contribute to the vertical force.) For the same condition the theoretical value of the reflection coefficient is 0.997, giving a predicted second-harmonic vertical force from the above equation of 18.4 N/m. This nearly perfect agreement suggests the utility of the present theory.

More extensive comparisons can be made with the computations performed by

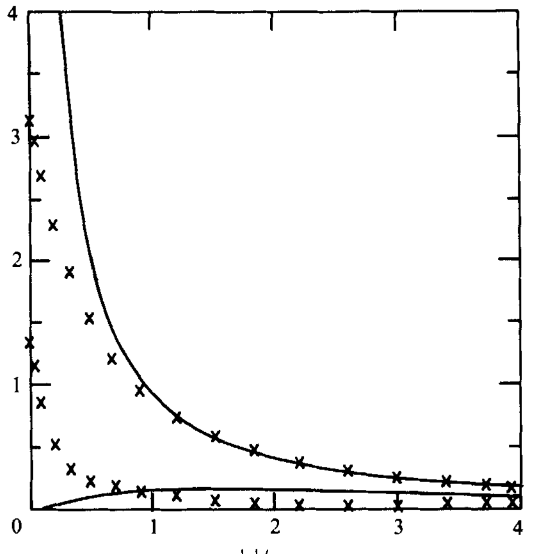

FIGURE 1. Approximation for the second-order, second-harmonic pressure distribution on a vertical axisymmetric cylinder as a function of depth, on the weather side ($\theta = \pi$, upper curve) and the lee side ($\theta = 0$, lower curve). These results are for a cylinder of draught $4a$, where a is the radius, and $Ka = 1.52$. The pressure is normalized by the quantity $\rho g A^2/a$. The crosses denote the corresponding results from the numerical solution by Kim & Yue (1988).

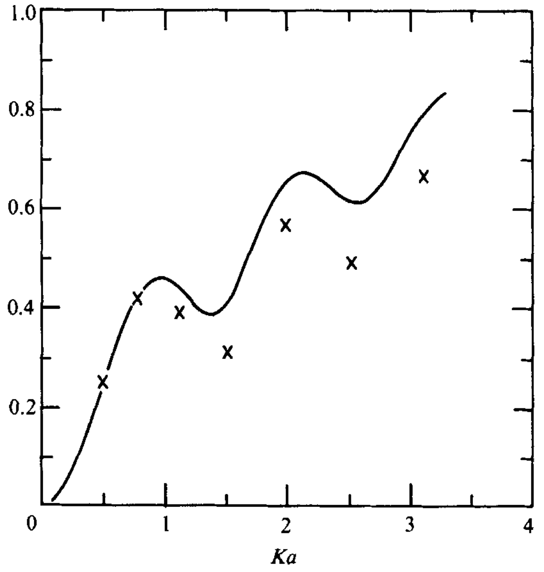

FIGURE 2. Second-order, second-harmonic vertical force on the cylinder (see figure 1). The force is normalized by the quantity $\rho g a A^2$. The crosses denote the corresponding results from the numerical solution by Kim & Yue (1988).

Kim & Yue (1988). These are for the second-order distribution and vertical force acting on a circular cylinder of radius a and draught $d = 4a$, in a fluid of finite depth equal to twice the cylinder draught.

Since the asymptotic analysis is based on the assumption of large draught, the

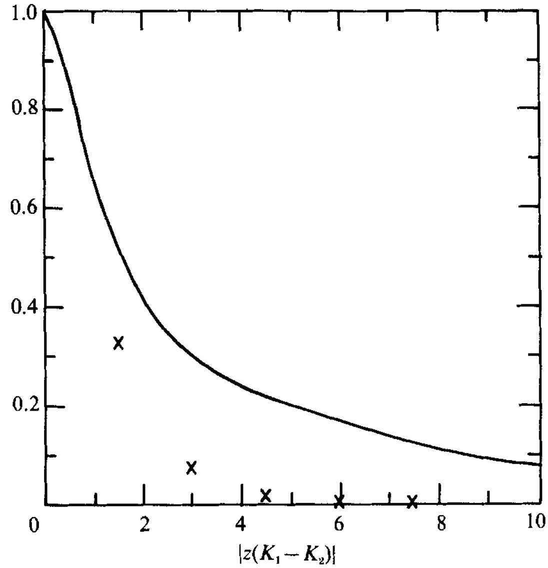

FIGURE 3. Attenuation factor for the sum-frequency pressure on the cylinder axis, in the presence of two incident waves with wavenumbers K_1, K_2 and frequencies ω_1, ω_2, as a function of the wavenumber difference and depth z. The cylinder draught is $4a$, and the sum of the two frequencies is fixed at the non-dimensional value $(\omega_1+\omega_2)(a/g)^{\frac{1}{2}} = 2.11$. The crosses denote the corresponding results for the vertical pressure force acting on the bottom of the cylinder, based on the numerical solution by Kim & Yue (1988).

first-order scattering function $f(\theta)$ can be evaluated from the analytic solution for a bottom-mounted cylinder (cf. Mei 1983). It follows that

$$F(\pi) = -4\omega K A^2 \left(\frac{2}{\pi}\right)^{\frac{1}{2}} e^{i\pi/4} \sum_{n=0}^{\infty} \epsilon_n (-)^n \frac{J_n'(Ka)}{H_n'(Ka)}, \tag{27}$$

where $H_n = J_n - iY_n$ is the Hankel function of the second kind and primes denote differentiation with respect to the argument.

Figure 1 shows the vertical distribution of the second-order pressure along the weather ($\theta = \pi$) and lee ($\theta = 0$) sides of the cylinder, including both the approximate result based on (19) and numerical computations of Kim & Yue (1988). (Only the component evaluated by Kim & Yue from the inhomogeneous free-surface boundary condition is included, since this is dominant in the regime where the present approximations are useful.) The comparison is satisfactory on the weather side of the cylinder, where the second-order pressure is relatively large.

Another example is based on the use of (21) to evaluate the vertical pressure force on the bottom of the cylinder. Only the term with $n = 0$ contributes, and after evaluating the pressure from Bernoulli's equation it follows that

$$\begin{aligned} F_z &= -2i\omega\rho \int_0^{2\pi} d\theta \int_0^a \phi_2(r,\theta)\, r\, dr \\ &= \frac{i\omega\rho}{K^2}\left(\frac{\pi}{2}\right)^{\frac{1}{2}} F(\pi)\, e^{i\pi/4} \int_0^a \frac{r\, dr}{(r^2+d^2)^{\frac{1}{2}}} \\ &= \frac{i\omega\rho}{K^2}\left(\frac{\pi}{2}\right)^{\frac{1}{2}} F(\pi)\, e^{i\pi/4}\, [(a^2+d^2)^{\frac{1}{2}} - d]. \end{aligned} \tag{28}$$

The result is shown in figure 2, and compared to the corresponding computations made by Kim & Yue. The agreement is precise for the two lowest frequencies, but at higher frequencies the approximate results exceed the numerical computations by 10–20 %. Despite this small difference there is a striking agreement in the frequency dependence, confirming the importance of the far-field scattering amplitude factor $F(\pi)$.

Finally, we show in figure 3 the magnitude of the attenuation factor for the sum-frequency vertical force, based on the approximation (26) for the pressure on $r = 0$. (The result shown here is based on forming a symmetric pressure matrix from (26), $i\rho(\omega_j+\omega_l)[\phi_2(\omega_j,\omega_l)+\phi_2(\omega_l,\omega_j)]$, and normalizing this pressure by the limiting value when $\omega_j = \omega_l$.) Also shown in this figure are the corresponding ratios of the vertical pressure forces computed by Kim & Yue. As the difference in frequencies and wavenumbers increases, the approximation based on (26) decreases algebraically, as anticipated above. On the other hand, the numerical results appear to decrease exponentially, suggesting that the approximation (26) is useful only for small values of δ. It is clear that (26) is not a consistent asymptotic approximation, for larger values of δ, since its order of magnitude, $O(z^{-2})$, is comparable with the neglected terms in (16) and (17).

7. Discussion

The approximation which has been derived leads to a simple description of the second-harmonic pressure field in the neighbourhood of a two- or three-dimensional diffracting body. The essential steps in the analysis are to consider only the contribution to the inhomogeneous free-surface boundary condition from the far-field approximation of the quadratic forcing function, and to integrate this effect over the entire free surface. The result is shown to be a consistent asymptotic approximation, to leading order. In the two-dimensional case the pressure field near the body is equal to half of the corresponding constant pressure in the partial standing wave upstream. In the three-dimensional case the second-harmonic pressure is inversely proportional to depth, with a magnitude proportional to the first-order scattered wave in the direction opposite to the incident-wave propagation.

The second-order potential which has been derived here is a particular solution, intended only to satisfy Laplace's equation and the inhomogeneous free-surface boundary condition in the far field. To this solution should be added a homogeneous component so as to satisfy the homogeneous Neumann boundary condition on the body surface. However, the particular solution varies slowly in space, and its spatial gradient is one order of magnitude smaller than the potential itself; thus the pressure field is not affected to leading order by the additional homogeneous component of the solution. In effect, the particular solution leads to a pressure field not unlike the hydrostatic component, with little effect on the integrated force for small bodies (or subelements of structures) which are totally submerged. The practical effects are much greater for floating bodies, particularly if they are restrained in the vertical direction.

No regard has been given to the radiation condition of outgoing waves at infinity. Similarly, integral transforms have been used to derive the solution without considering the integrability of the quadratic forcing function q. A possible justification is to assume that the frequency ω and wavenumber $K = \omega^2/g$ are complex, with a vanishingly small imaginary component which is negative. This modification of the physical parameters is consistent with the requirement of an initial state of rest at $t = -\infty$, and it also renders the quadratic forcing function asymptotically small as $r \to \infty$, thus overcoming any questions regarding the interpretation of (6) or (14). However, this modification results in an incident-wave potential (2) with exponentially large amplitude at $x = -\infty$; one can argue that this is irrelevant within a finite domain including the diffracting body, and that such a disturbance is physically realizable in a finite wave tank. A more detailed mathematical justification is given by Wang (1987), within the framework of an initial-value problem.

Comparisons with limited experimental data and more complete numerical solutions indicate that our approximation is useful in describing the second-harmonic vertical force on a two-dimensional cylinder, and the dominant part of the pressure distribution and vertical acting on a three-dimensional axisymmetric cylinder at depths comparable with or larger than the radius. It remains to be shown that useful predictions can also be made for non-compact bodies, such as tension-leg platforms, where exact numerical predictions are not available for comparison.

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REFERENCES

EATOCK TAYLOR, R. & HUNG, S. M. Second order diffraction forces on a vertical cylinder in regular waves. *Appl. Ocean Res.* **9**, 19–30.

GRADSHTEYN, I. S. & RYZHIK, I. M. 1965 *Table of Integrals, Series, and Products.* Academic.

JOHANSSON, M. 1989 Barrier-type breakwaters – transmission, reflection and forces. Ph.D. thesis, Chalmers University of Technology.

KIM, M. H. & YUE, D. K. P. 1988 The nonlinear sum-frequency wave excitation and response of a tension-leg platform. *5th Intl Conf. on the Behaviour of Offshore Structures, BOSS, Trondheim*, pp. 687–704.

KIM, M. H. & YUE, D. K. P. 1989 The complete second-order diffraction solution for an axisymmetric body. Part 1. Monochromatic incident waves. *J. Fluid Mech.* **200**, 235–264.

MEI, C. C. 1983 *The Applied Dynamics of Ocean Surface Waves.* Wiley-Interscience.

MOLIN, B. 1979 Second order diffraction loads upon three-dimensional bodies. *Appl. Ocean Res.* **1**, 197–202.

MOLIN, B. 1986 Second-order frequency loads and motions for 3D bodies. *Proc. First Intl Workshop on Water Waves and Floating Bodies, Dept. of Ocean Engng, MIT, Rep.* 86–2, pp. 109–113.

NEWMAN, J. N. 1977 *Marine Hydrodynamics.* MIT Press.

OGILVIE, T. F. 1983 Second-order hydrodynamic effects on ocean platforms. *Proc. Intl Workshop on Ship and Platform Motions, Dept. of Naval Arch. & Offshore Engng, University of California, Berkeley.*

SHIMADA, K. 1987 Solutions to three-dimensional second-order diffraction problems by means of simple-source integral-equation method. *Soc. Nav. Arch. Japan*, **161**, 139–145.

WANG, P. F. 1987 The radiation condition and numerical aspects of second-order surface wave radiation and diffraction. Ph.D. thesis, MIT.

WEHAUSEN, J. V. & LAITONE, E. V. 1960 Surface waves. *Handbuch der Physik*, vol. 9, pp. 445–778.